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THE APPLICATIONS OF CG AND PCG ON ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS SOLVED BY FINITE ELEMENT METHOD

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Abstract

We apply the CG and PCG methods to the linear system $Hx = b$. which was derived from elliptic partial differential equation by using finite element method in order to get the number of iteration and residual. Two examples will be used for implications.

Key words

CG and PCG methods, Elliptic partial differential equations, finite element methods.

Introduction

Let Ω be a bounded domain in R^d, with boundary $\partial\Omega$.

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It is assumed that f and g are continuous on Ω. So a unique solution exists

(1)

$$-\Delta u = f(x, y) \qquad \text{For } (x, y) \in \Omega \text{ and}$$
$$u(x,y)=g(x,y) \qquad \text{in } \partial\Omega$$
$$\Omega = \{(x, y): a < x < b, c < y < d\}.$$

for the basis of the finite element approximation of (1). First the test function v is picked where v satisfies the boundary condition $v = 0$ on $\partial\Omega$. Multiply the first equation by v, and integrate the equation over Ω, and the Green's formula, is used

$$\int_{\Omega} -\Delta u\, v\, dx = -\int_{\partial\Omega} \frac{\partial u}{\partial n} v\, ds + \int_{\Omega} \nabla u\, \nabla v\, dx,$$

A finite element discretization of (1) is based on the weak formulation: seek $u \in V = [H_0^1(\Omega)]^d$ such that

(2)

$$a(u, v) + b(v, p) = (f, v) \quad \forall v \in V$$

where $a(u, v) = \int_{\Omega} \nabla u\, \nabla v\, dx$ ، $(f, v) = \int_{\Omega} f\; v\, dx$ ·

the approximate solution $u_h \in V_h \subset V$ satisfies

(3)

$$a(u_h, v) = (f, v) \quad \forall v \in V_h$$

If $u_h = \sum_{i=1}^{N} u_i \phi_i(x)$ and substitutes them in the equations

$$a(u_h, \phi_k) = (f, \phi_k) \qquad \forall\, k = 1,\dots, N$$

It results in

$$\sum_{i=1}^{N} u_i . a(\phi_i, \phi_k) = (f, \phi_k) \qquad \forall\, k = 1,\dots., N$$

It can be rewritten in the following form

(4)

$$Hx = b \cdot$$

Here $H \in R^{n\times n}$ is the symmetric matrix $b \in R^n$, and for the solution $x \in R^n$ is obtained

In this paper we will use conjugate gradient method and preconditioned conjugate gradient methods to solve the system of linear equations.

$$Hx = b$$

[1] Conjugate gradient method. (CG)

We will use the conjugate gradient method to solve the system of linear algebraic equations

$$Hx = b \cdot \qquad (1.1)$$

We choose an initial approximation x_0, put

$$r_0 = b - Hx_0, \qquad p_0 = r_0 \qquad (1.2)$$

and compute

$$a_j = \frac{r_j^T r_j}{p_j^T H p_j},$$

$$x_{j+1} = x_j + a_j p_j$$

$$r_{j+1} = r_j - a_j H p_j, \qquad (1.3)$$

$$b_j = \frac{r_{j+1}^T r_{j+1}}{r_j^T r_j},$$

$$p_{j+1} = r_{j+1} + b_j p_j.$$

[2] Preconditioned Conjugate gradient method. (PCG)

Let us denote B as the preconditioner of the symmetric positive definite matrix where the matrix B is closed to the matrix H of the system.

If we put

$$H=B-E \qquad (1.4)$$

Then we can, e.g., require that the norm $\|E\| = \|B-H\|$ to be small.

It is well known that for every symmetric positive definite matrix B, there exists just one symmetric positive definite matrix $B^{-\frac{1}{2}}$ (the square root of B^{-1}) such that

$$B^{-1} = B^{-\frac{1}{2}} B^{-\frac{1}{2}}.$$

Thus, put

$$\hat{b} = B^{-\frac{1}{2}} b \qquad (1.5)$$

and rewrite the system (1) as

$$\hat{H}\hat{x} = \hat{b}, \tag{1.6}$$

where

$$\hat{H} = B^{-\frac{1}{2}} H B^{-\frac{1}{2}}, \tag{1.7}$$

$$\hat{x} = B^{\frac{1}{2}} x. \tag{1.8}$$

We now can solve the system (1.6) using the conjugate gradient method (1.2), (1.3) and the solution x computed from (1.8).

Notes that, according to (1.4) and (1.7),

$\hat{H} = I - B^{-\frac{1}{2}} E B^{-\frac{1}{2}} = I - \hat{E}$, where we put $\hat{E} = B^{-\frac{1}{2}} E B^{-\frac{1}{2}}$.

After substituting $\hat{H}$, $\hat{x}$ and $\hat{b}$ from the formulae (1.5), (1.7) and (1.8), the algorithm (1.2) and (1.3) for solving the system (1.6) can be rewritten as follows. Choose the initial approximation x_0, put

$$r_0 = B^{-\frac{1}{2}} b - B^{-\frac{1}{2}} H B^{-\frac{1}{2}} B^{\frac{1}{2}} x_0, \qquad p_0 = r_0,$$

and compute

$$a_j = \frac{r_j^T r_j}{p_j^T B^{-\frac{1}{2}} H B^{-\frac{1}{2}} p_j},$$

$$B^{\frac{1}{2}} x_{j+1} = B^{\frac{1}{2}} x_j + a_j p_j,$$

$$r_{j+1} = r_j - a_j B^{-\frac{1}{2}} H B^{-\frac{1}{2}} p_j,$$

$$b_j = \frac{r_{j+1}^T r_{j+1}}{r_j^T r_j},$$

$$p_{j+1} = r_{j+1} + b_j p_j.$$

Substituting $\hat{r}_j = B^{\frac{1}{2}} r_j$ and $\hat{p}_j = B^{-\frac{1}{2}} p_j$ we finally get the preconditioned algorithm (instead of $\hat{r}_j$ and $\hat{p}_j$ we write r_j and p_j):

Choose the initial approximation x_0, put

$$r_0 = b - Hx_0, \qquad p_0 = B^{-1} r_0,$$

and compute

$$a_j = \frac{r_j^T B^{-1} r_j}{p_j^T H p_j},$$

$$x_{j+1} = x_j + a_j p_j,$$

$$r_{j+1} = r_j - a_j H p_j,$$

$$b_j = \frac{r_{j+1}^T B^{-1} r_{j+1}}{r_j^T B^{-1} r_j},$$

$$p_{j+1} = B^{-1} r_{j+1} + b_j p_j.$$

How to choose starting and stopping points?

If you have a rough estimate of the value of x, use it as the starting value x_0. If not, set $x_0 = 0$. When CG reaches the minimum point, the residual becomes zero; we must stop immediately when the residual is zero.

The preconditioners

[1] The preconditioner of tridiagonal of the matrix H (B_T .)

The matrix H is preconditioned by the preconditioner matrix $B_T = \text{tridiag}(H)$ and $T_T = B_T^{-1}H$ is the preconditioned matrix.

[2]The preconditioner of incomplete LU factorization of the matrix H (B_{ILU})

The matrix H is preconditioned by the preconditioner matrix B_{ILU} where L is a lower triangular matrix and U is an upper triangular matrix and $T_{ILU} = B_{ILU}^{-1}H$ is the preconditioned matrix.

Results and Applications

We will use two examples for the conjugate gradient method and preconditioned conjugate gradient method

Example 1

Consider the problem in the given domain with boundary conditions:

$$-\Delta u = 2 - (x^2 + y^2) \qquad 0 \le x, y \le 1$$
$$u(x, y) = 0 \qquad \text{in boundary}$$

Example 2

Consider the problem in the given domain with boundary conditions

$$-\Delta u = e^{-x}(x - 2 + y^3 + 6y) \qquad 0 \le x, y \le 1$$
$$u(x, y) = 0 \qquad \text{in boundary}$$

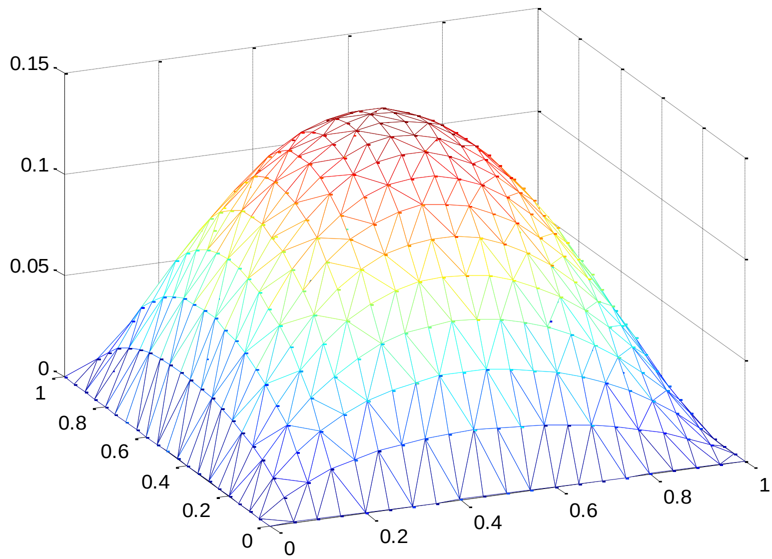

Figure 1.1 The shape of example 1 where n = 20

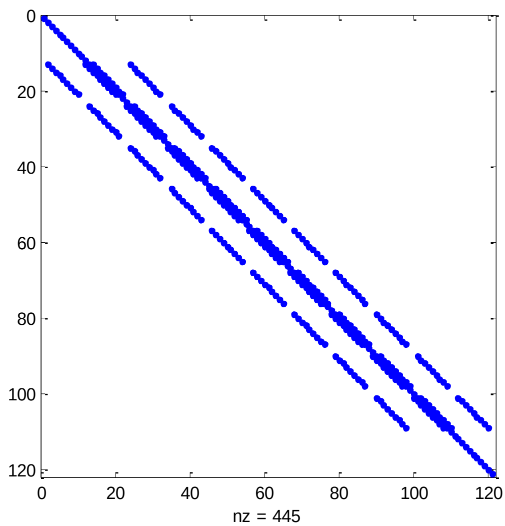

Figure 1.2 The final shape of matrix H of example 1 where n = 10

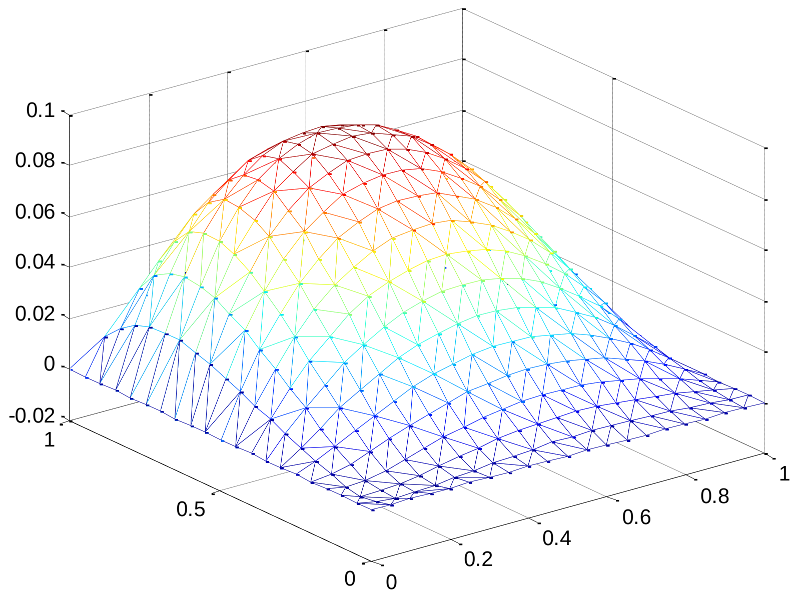

Figure 1.3 The shape of example 2 where n = 20

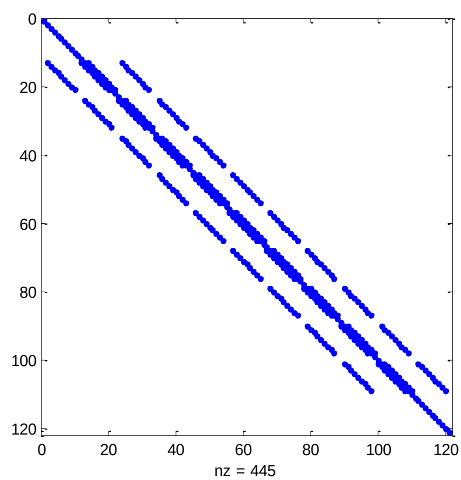

Figure 1.4 The final shape of matrix H of example 2 where n = 10

The results of numerical tests of the conjugate gradient method and preconditioned conjugate gradient method for example 1 and example 2 are given in the following tables:

Table 1 Iteration and residual of matrix H by the conjugate gradient method

of example 1

h		1/10	1/20	1/30	1/40
CG	Residual	6.4411e-4	9.2641e-4	8.7947e-4	8.6261e-4
	Iteration	12	20	26	28

Table 2 Iteration and residual of matrix H by the preconditioned conjugate

gradient method of example 1

h		1/10	1/20	1/30	1/40
T_T	Residual	5.4801e-4	7.7129e-4	8.8534e-4	8.0969e-4
	Iteration	9	15	19	22
T_{ILU}	Residual	3.2854e-4	7.4100e-4	1.5134e-4	1.4968e-4
	Iteration	3	2	3	2

Table 3 Iteration and residual of matrix H by the conjugate gradient method

of example 2

h		1/10	1/20	1/30	1/40
CG	Residual	7.4896e-4	9.9616e-4	9.7013e-4	9.6951e-4
	Iteration	12	22	29	37

Table 4 Iteration and residual of matrix H by the

preconditioned conjugate

gradient method of example 2

h		1/10	1/20	1/30	1/40
T_T	Residual	6.4889e-4	7.3937e-4	9.3371e-4	7.7787e-4
	Iteration	9	16	21	28
T_{ILU}	Residual	3.4696e-4	2.9969e-4	4.7331e-5	1.2640e-4
	Iteration	3	2	2	2

Conclusion. It is shown in this paper that in the CG method the number of iterations increases when the size of matrix H increases. The convergent in the preconditioned matrix is faster than in the non preconditioned one. The best preconditioner is the incomplete preconditioner.

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